

Constant and Equivariant Cyclic Cohomology

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Abstract

In this note we prove that the constant and equivariant cyclic cohomology of algebras coincide. This shows that constant cyclic cohomology is rich and computable.

1 Introduction

Cyclic cohomology, invented by A. Connes [6] as a cornerstone of noncommutative geometry, is the noncommutative analogue of de Rham homology. It plays a crucial role in noncommutative geometry by detecting topological invariants such as K -theory of algebras via a pairing which is a nontrivial generalization of the index of operators in the classical geometry. There is a variety of specialized cyclic cohomology theories, which are defined in various contexts for different purposes. In this paper we briefly discuss two of these theories and prove that they are actually the same.

Symmetry in classical geometry is defined via the actions of groups on spaces. In noncommutative geometry spaces are replaced with not necessarily commutative algebras, viewed as algebras of coordinates of "noncommutative spaces". Accordingly, the role of symmetry is played by actions of Hopf

algebras on algebras. A natural entry into this page of the ‘commutative-noncommutative’ dictionary is the counterpart of equivariant de Rahm homology, which evidently is the equivariant cyclic cohomology. Equivariant cyclic cohomology has been studied by various authors, in particular we refer the reader to [3, 2, 8, 9, 12, 13], for the case of groups, and [1, 11, 10], for Hopf algebras.

In homological algebra, it is often important to replace a complex with a quasiisomorphic subcomplex. In cyclic cohomology it is known that normalized co-chains form a subcomplex quasiisomorphic to cyclic complex. The former are those co-chains that vanish on the image of degeneracies. Restricting on the normalized subcomplex is usually useful and sometime even necessary, for example in the proof of the cyclic Eilenberg-Zilber theorem. Connes in [4] in the study of spectral triple over $A(SU_q(2))$ generalized the normalized cyclic complex to constant cyclic subcomplex. Let us briefly recall it. Let \mathcal{C} be a unital subalgebra of a unital algebra \mathcal{D} . A \mathcal{C} -constant co-chain is a co-chain $\varphi \in \text{Hom}(D^{\otimes(n+1)}, \mathbb{C})$, such that $\varphi(a^0, \dots, a^n)$, and $b\varphi(a^0, \dots, a^{n+1})$ vanish whenever $a^j \in \mathcal{C}$, for at least one $j \geq 1$. Then the \mathcal{C} -constant co-chains form a sub-complex of the mixed-complex computing the cyclic cohomology of \mathcal{D} and hence one defines \mathcal{C} -constant cyclic cohomology of \mathcal{D} as the cohomology of this subcomplex.

Now let us consider a Hopf algebra \mathcal{H} , with invertible antipode, acting on an algebra A and makes A an \mathcal{H} -module algebra. One knows that the algebra \mathcal{H} is a subalgebra of the crossed product algebra $A \rtimes \mathcal{H}$. So we can talk about \mathcal{H} -constant cohomology of $A \rtimes \mathcal{H}$. On the other hand equivariant cyclic cohomology of A under the action of \mathcal{H} is well-defined. In this paper we prove that the two theories coincide.

Throughout this note we assume that algebras are unital and associative, that every algebraic structure including Hopf algebras are defined over \mathbb{C} , and that Hopf algebras have invertible antipode. The antipode, coproduct and counit of Hopf algebras are denoted by S , Δ , and ϵ respectively. The action of Δ on an element of \mathcal{H} , say h , is denoted by $h_{(1)} \otimes h_{(2)}$.

2 Constant Cyclic Cohomology

Let A be a unital algebra. We recall, from [6], the cyclic cohomology of A is the cohomology of the space of all cyclic co-chains, which are those Hochschild co-chains that are cyclic, i.e, $f \in \text{Hom}(A^{\otimes(n+1)}, \mathbb{C})$ satisfy $f(a^0, \dots, a^n) =$

$(-1)^n f(a^n, a^0 \dots, a^{n-1})$. The space of all cyclic co-chains forms a subcomplex of Hochschild complex of the algebra A with coefficients in A^* , the dual space of A .

Cyclic cohomology of A can be also defined via the Connes (b, B) -bicomplex. Let us recall this bicomplex. let $\mathcal{B}^{n,m} = \mathcal{B}^{n,m}(A) = A^{\otimes(n-m)}$ and 0 where $m > n$ or $m < 0$. With the Connes boundary map $B : \mathcal{B}^{n,m} \rightarrow \mathcal{B}^{n,m+1}$, and the Hochschild's one $b : \mathcal{B}^{n,m} \rightarrow \mathcal{B}^{n+1,m}$, \mathcal{B} forms a bicomplex. The cyclic complex of A is quasisomorphic to the total complex of $\mathcal{B}(A)$. The vertical and horizontal boundary maps are defined as follows.

First $B := AB_0$, where

$$(B_0\varphi)(a^0, \dots, a^n) := \varphi(1, a^0, a^1, \dots, a^n) - (-1)^{n+1} \varphi(a^0, a^1, \dots, a^n, 1) \quad (2.1)$$

$$A\varphi(a^0, a^1, \dots, a^n) := \sum_0^n (-1)^{nj} \varphi(a^j, a^{j+1}, \dots, a^{j-1}), \quad (2.2)$$

and the Hochschild boundary map,

$$b\varphi(a^0, \dots, a^{n+1}) := \quad (2.3)$$

$$\sum_0^n (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, a^1, \dots, a^n). \quad (2.4)$$

Let us recall the definition of constant cyclic cohomology from [4]. Let $\mathcal{C} \subset \mathcal{D}$ be a unital subalgebra of a unital algebra \mathcal{D} . We say a co-chain $\varphi \in \text{Hom}(\mathcal{D}^{\otimes(n+1)}, \mathbb{C})$ is \mathcal{C} -constant if and only if both $\varphi(a^0, \dots, a^n)$, and $b\varphi(a^0, \dots, a^{n+1})$ vanish whenever at least one of the a^j , $j \geq 1$, is in \mathcal{C} .

Since the subalgebra \mathcal{C} is unital one sees that $B_0\varphi(a^0, \dots, a^n) = \varphi(1, a^0, \dots, a^n)$, for a \mathcal{C} -constant co-chain φ . This proves that the boundary B sends \mathcal{C} -constant co-chains to themselves. So \mathcal{C} -constant co-chains form a subcomplex of the (b, B) bicomplex and hence we can define the \mathcal{C} -constant cyclic cohomology to be the total cohomology of this subcomplex. We denote this cohomology by $HC^*(\mathcal{D}; \mathcal{C})$.

If one takes the ground field as the subalgebra \mathcal{C} then the \mathcal{C} -constat co-chains are just normalized co-chains and hence $HC^*(\mathcal{D}; \mathbb{C}) = HC^*(\mathcal{D})$.

Now let \mathcal{H} be a Hopf algebra acting on an algebra A and makes A a \mathcal{H} -module algebra. The latter means that the multiplication and unit map of A are \mathcal{H} -linear, i.e, $h(ab) = h_{(1)}(a)h_{(2)}(b)$, and $h(1) = \epsilon(h)$. Having the above situation, we make a crossed product algebra $A \rtimes \mathcal{H}$, which has $A \otimes \mathcal{H}$ as its

underlying vector space, $1 \otimes 1$ as its unit, and $(a \otimes h)(b \otimes g) = ah_{(1)}(a) \otimes h_{(2)}g$ as its multiplication. One can see that \mathcal{H} is a unital subalgebra of $A \rtimes \mathcal{H}$, and hence \mathcal{H} -constant cyclic cohomology of $A \rtimes \mathcal{H}$ is well-defined. Our main goal in this paper is to compute this cohomology in terms of equivariant cyclic cohomology defined in [1, 11] which we recall it in the next section.

3 Equivariant Cyclic Cohomology

Cyclic cohomology in its ultimate extent is defined for any co-cyclic module [6, 5]. A co-cyclic module is a co-simplicial module $M = (\{M^n\}_{n \geq 0}, d_n^i, s_n^j)$, $0 \leq i \leq n+1$, $0 \leq j \leq n-1$, together with a cyclic operator $t_n : M^n \rightarrow M^n$, for each $n = 0, 1, \dots$ satisfy the following identities.

$$t_{n+1}d_n^i = d_n^{i-1}t_n, \quad , t_{n+1}d_n^0 = d_n, \quad (3.1)$$

$$t_{n-1}s_n^j = s_n^{j-1}t_n, \quad t_{n-1}s_n^0 = s_n^{n-1}t_n^2, \quad (3.2)$$

$$t_n^{n+1} = id \quad (3.3)$$

The cyclic cohomology of a co-cyclic module is defined to be the cohomology of total complex of $\mathcal{B}(M)$ which is a bicomplex that in degree (n, m) is M^{n-m} and 0 above the main diagonal and below the horizontal axis. Its vertical and horizontal boundaries B and b are defined as follows.

$$b := \sum_0^{n+1} (-1)^i d_n^i, \quad (3.4)$$

$$B := AB_0, \quad (3.5)$$

$$B_0 := (1 + (-1)^n t_n) s_n, \quad (3.6)$$

$$A := \sum_0^n (-1)^{nj} t_n^j \quad (3.7)$$

Now, let us recall the definition of equivariant cyclic cohomology. Equivariant cyclic cohomology of an algebra under the action of a discrete group is studied in [3, 2, 8, 9, 12, 13]. Its generalization for the action of a Hopf algebra on an algebra is dealt with in [1, 11, 10]. In the following we recall this cohomology theory.

Let $C_{\mathcal{H}}^n(A) = \text{Hom}_{\mathcal{H}}(\mathcal{H} \otimes A^{\otimes(n+1)}, \mathbb{C})$, which is all equivariant co-chains, i.e., all $\varphi : \mathcal{H} \otimes A^{\otimes(n+1)} \rightarrow \mathbb{C}$, such that for all $g, h \in \mathcal{H}$, and $a_i \in A$,

$$\varphi(g_{(1)} \cdot h, g_{(2)}(a^0), \dots, g_{(n+1)}(a^n)) = \epsilon(g)\varphi(h, a^0, \dots, a^n), \quad (3.8)$$

where $g \cdot h = g_{(1)}hS^{-1}(g_{(2)})$ is the usual adjoint action of \mathcal{H} on itself.

Lemma 3.1. *The condition (3.8) is equivalent to*

$$f(S(g_{(2)})hg_{(1)}, a^0, \dots, a^n) = f(h, g_{(1)}(a^0), \dots, g_{(n+1)}(a^n)). \quad (3.9)$$

Proof. Let f satisfy the condition (3.8).

$$\begin{aligned} f(h, g_{(1)}(a^0), \dots, g_{(n+1)}(a^n)) &= \\ \epsilon(g_{(1)})f(h, g_{(2)}(a^0), \dots, g_{(n+2)}(a^n)) &= \\ \epsilon(S(g_{(1)}))f(h, g_{(2)}(a^0), \dots, g_{(n+2)}(a^n)) &= \\ f(S(g_{(1)})_{(1)} \cdot h, S(g_{(1)})_{(2)}g_{(2)}(a^0), \dots, S(g_{(1)})_{(n+2)}g_{(n+2)}(a^n)) &= \\ f(S(g_{(2)})hg_{(1)}, a^0, \dots, a^n). \end{aligned}$$

The converse is similarly checked. \square

One can see that, see one of [1, 11, 10], $C_{\mathcal{H}}^*(A)$ with the following operators is a co-cyclic module.

$$d_n^i \varphi(h, a^0, \dots, a^{n+1}) = \varphi(h, a^0, \dots, a^i a^{i+1}, \dots, a^{n+1}), \quad 0 \leq i \leq n \quad (3.10)$$

$$d_n^{n+1} \varphi(h, a^0, \dots, a^{n+1}) = \varphi(h_{(2)}, S^{-1}(h_{(1)})(a^{n+1})a^0, a^1, \dots, a^n), \quad (3.11)$$

$$s_n^j \varphi(h, a^0, \dots, a^{n-1}) = \varphi(h, a^0, \dots, a^j, 1, a^{j+1}, \dots, a^{n-1}); 0 \leq j \leq n-1, \quad (3.12)$$

$$t_n \varphi(h, a^0, \dots, a^n) = \varphi(h_{(2)}, S^{-1}(h_{(1)})(a^n), a^0, \dots, a^{n-1}). \quad (3.13)$$

We recall a cyclic map between co-cyclic modules is a linear map which commutes with co-simplicial and cyclic operators.

Proposition 3.2. *The following map is cyclic.*

$$\Psi : C_{\mathcal{H}}^n(A) \longrightarrow C^n(A \rtimes \mathcal{H}), \quad (3.14)$$

$$\begin{aligned} \Psi(\varphi)(a^0 \otimes h^1, \dots, a^n \otimes h^n) &= \\ \varphi(h_{(n+1)}^0 \dots h_{(2)}^{n-1} h^n, a^0, h_{(1)}^0(a^1), \dots, h_{(n)}^0 \dots h_{(1)}^{n-1}(a^n)). \end{aligned}$$

Proof. We need to show that Ψ commutes with cyclic structure of both hand sides. The only non trivial ones are the commutation with cyclic operators and the last face maps. We only show the former and leave the rest to the reader. Let τ_n be the cyclic operator of the co-cyclic module $C^*(A \rtimes \mathcal{H})$. Indeed,

$$\begin{aligned}
& \tau_n(\Psi(\varphi))(a^0 \otimes h^1, \dots, a^n \otimes h^n) = \\
& \Psi(\varphi)(a^n \otimes h^n, a^0 \otimes h^0, \dots, a^{n-1} \otimes h^{n-1}) = \\
& \varphi(h^n_{(n+1)} h^0_{(n)} \dots h^{n-2}_{(2)} h^{n-1}, a^n, h^n_{(1)}(a^0), \dots, h^n_{(n)} h^0_{(n-1)} \dots h^{n-2}_{(1)}(a^{n-1})) = \\
& \varphi(h^n_{(n+1)} \cdot (h^0_{(n)} \dots h^{n-2}_{(2)} h^{n-1} h^n_{(n+2)}), a^n, h^n_{(1)}(a^0), \dots \\
& \dots, h^n_{(n)} h^0_{(n-1)} \dots h^{n-2}_{(1)}(a^{n-1})) = \\
& \varphi(h^0_{(n)} \dots h^{n-2}_{(2)} h^{n-1} h^n_{(n+2)}, S^{-1}(h^n_{(n+1)}) \cdot (a^n, h^n_{(1)}(a^0), \dots \\
& \dots, h^n_{(n)} h^0_{(n-1)} \dots h^{n-2}_{(1)}(a^{n-1}))) = \\
& \varphi(h^0_{(n)} \dots h^{n-2}_{(2)} h^{n-1} h^n_{(2)}, S^{-1}(h^n_{(1)})(a^n), a^0, h^0_{(1)}(a^1), \dots \\
& \dots, h^0_{(n-1)} \dots h^{n-2}_{(1)}(a^{n-1})) = \\
& \Psi(t\varphi)(a^0 \otimes h^0, \dots, a^n \otimes h^n).
\end{aligned}$$

□

We are ready now to determine the image of Ψ . The proof of the following proposition is obvious.

Proposition 3.3. *The map Ψ lands in $C^*(A \rtimes \mathcal{H}; \mathcal{H})$ provided we restrict Ψ to the normalized subcomplex of $C^*_\mathcal{H}(A)$.*

Theorem 3.4. *The restriction of Ψ to the normalized co-chains is an isomorphism and hence*

$$HC^*_\mathcal{H}(A) \cong HC^*(A \rtimes \mathcal{H}; \mathcal{H}). \quad (3.15)$$

Proof. Let us by abuse of notation denote the restriction of Ψ on the normalized complex again by Ψ . Using the above propositions, to finish the proof we need the inverse of Ψ . Let $\Phi : C^*(A \rtimes \mathcal{H}; \mathcal{H}) \rightarrow \bar{C}^*_\mathcal{H}(A)$, where \bar{C} denotes the normalized complex, be defined by:

$$(\Phi f)(h, a^0, \dots, a^n) = f(a^0 \otimes 1, \dots, a^{n-1} \otimes 1, a^n \otimes h).$$

First we need to show that Φ is well-defined. To this end we check that $\Phi(f)$ is equivariant if f is \mathcal{H} -constant. Let f be \mathcal{H} -constant. So,

$$bf(a^0 \otimes h^0, 1 \otimes h, a^2 \otimes h^2, \dots, a^n \otimes h^n) = 0.$$

We conclude that,

$$\begin{aligned} f(a^0 \otimes h^0 h, a^2 \otimes h^2, \dots, a^n \otimes h^n) = \\ f(a^0 \otimes h^0, h_{(1)}(a^1) \otimes h_{(2)}h^1, a^2 \otimes h^2, \dots, a^n \otimes h^n). \end{aligned} \quad (3.16)$$

Similarly, for each $i \leq n-1$, we have the following identity,

$$\begin{aligned} f(a^0 \otimes h^0, \dots, a^i h \otimes h^i, \dots, a^n \otimes h^n) = \\ f(a^0 \otimes h^0, \dots, h_{(1)}(a^{i+1}) \otimes h_{(2)}h^{i+1}, a^2 \otimes h^2, \dots, a^n \otimes h^n) \end{aligned} \quad (3.17)$$

And eventually we have,

$$\begin{aligned} f(a^0 \otimes h^0, \dots, a^{n-1} \otimes h^{n-1}, a^n \otimes h^n h) = \\ f(h_{(1)}(a^0) \otimes h_{(2)}h^0, a^1 \otimes h^1, \dots, a^n \otimes h^n). \end{aligned} \quad (3.18)$$

Now, by using (3.16), (3.17), (3.18), and Lemma 3.1, we show Φf is equivariant,

$$\begin{aligned} (\Phi f)(h, g_{(1)}(a^0), \dots, g_{(n+1)}(a^n)) = \\ f(g_{(1)}(a^0) \otimes 1, \dots, g_{(n)}(a^{n-1}) \otimes 1, g_{(n+1)}(a^n) \otimes h) = \\ f(g_{(1)}(a^0) \otimes 1, \dots, g_{(n)}(a^{n-1}) \otimes 1, g_{(n+1)}(a^n) \otimes g_{(n+2)}S(g_{(n+3)})h) = \\ f(g_{(1)}(a^0) \otimes 1, \dots, g_{(n)}(a^{n-1}) \otimes g_{(n+1)}, a^n \otimes S(g_{(n+3)})h) = \\ \vdots \\ f(g_{(1)}(a^0) \otimes g_{(2)}, a^1 \otimes 1, \dots, a^{n-1} \otimes 1, a^n \otimes S(g_{(3)})h) = \\ f(a^0 \otimes 1, \dots, a^{n-1} \otimes 1, a^n \otimes S(g_{(2)})hg_{(1)}) = \Phi f(S(g_{(2)})hg_{(1)}, a^0, \dots, a^n). \end{aligned}$$

The only thing left to finish the proof is to show that Ψ and Φ are inverse to one another. It is obvious that $\Phi\Psi = id$. Again by using (3.16), (3.17), and

(3.18), we show the $\Psi\Phi = id$.

$$\begin{aligned}
& \Psi\Phi f(a^0 \otimes h^0, \dots, a^n \otimes h^n) = \\
& \Phi f(h^0_{(n+1)} \dots h^{n-1}_{(2)} h^n, a^0, h^0_{(1)}(a^1), \dots, h^0_{(n)} \dots h^{n-1}_{(1)}(a^n)) = \\
& f(a^0 \otimes 1, h^0_{(1)}(a^1) \otimes 1, \dots, h^0_{(n-1)} \dots h^{n-2}_{(1)}(a^{n-1}) \otimes 1, \\
& h^0_{(n)} \dots h^{n-1}_{(1)}(a^n) \otimes h^0_{(n+1)} \dots h^{n-1}_{(2)} h^n) = \\
& f(a^0 \otimes h^0, a^1 \otimes 1, h^1_{(1)}(a^2) \otimes 1 \dots, h^1_{(n-2)} \dots h^{n-2}_{(1)}(a^{n-1}) \otimes 1, \\
& h^1_{(n-1)} \dots h^{n-1}_{(1)}(a^n) \otimes h^1_{(n)} \dots h^{n-1}_{(2)} h^n) = \\
& f(a^0 \otimes h^0, a^1 \otimes h^1, a^2 \otimes 1, h^2_{(1)}(a^3) \otimes 1, \dots, h^2_{(n-3)} \dots h^{n-2}_{(1)}(a^{n-1}) \otimes 1, \\
& h^2_{(n-2)} \dots h^{n-1}_{(1)}(a^n) \otimes h^2_{(n-1)} \dots h^{n-1}_{(2)} h^n) = \\
& \vdots \\
& f(a^0 \otimes h^0, \dots, a^n \otimes h^n)
\end{aligned}$$

□

It is shown that the equivariant cyclic cohomology and cyclic cohomology of crossed product algebra coincide if \mathcal{H} is semisimple [1].

Corollary 3.5. *If \mathcal{H} is semisimple, then $HC^*(A \rtimes \mathcal{H}; \mathcal{H}) \cong HC^*(A \rtimes \mathcal{H})$.*

Now let B be a unital sub \mathcal{H} -module algebra of A . That is, B is a unital subalgebra of A and $h(b) \in B$, for any $h \in \mathcal{H}$ and $b \in B$. Similarly with the same proof as of Theorem 3.4 one can show that

$$HC^*_{\mathcal{H}}(A; B) \cong HC^*(A \rtimes \mathcal{H}; B \rtimes H),$$

where the left hand side is the cyclic cohomology of subcomplex of B -constant equivariant co-chains; i.e, $\varphi \in \text{Hom}_{\mathcal{H}}(\mathcal{H} \otimes A^{\otimes(n+1)}, \mathbb{C})$, such that $\varphi(h, a^0, \dots, a^n)$, and $b\varphi(h, a^0, \dots, a^{n+1})$ vanish if at least one of a^j is in B , for $j \geq 1$.

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